Definitions and key facts for section 4.3

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. Otherwise, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent**, that is, if there exist weights $c_1, c_2, \ldots, c_p$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

In this case, we call (1) a **linear dependence relation**.

Fact: Characterization of linearly dependent sets A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

Indeed, if $\mathbf{v}_1 \neq \mathbf{0}$ and S is linearly dependent then some $\mathbf{v}_j$ is a linear combination of the *preceding* vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

So, a pair of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if one is a multiple of the other, and linearly independent if neither is a multiple of the other.

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

1. $\mathcal{B}$ is a linearly independent set, and
2. the subspace spanned $\mathcal{B}$ is H; that is

$$H = \operatorname{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\} = \operatorname{Span}\mathcal{B}.$$

Standard Bases:
The **standard basis** for $\mathbb{R}^n$ is the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$ containing the columns of I_n. Recall

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\\\vdots\\0\end{bmatrix}, \quad \cdots \quad , \mathbf{e}_n = \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}.$$

The **standard basis** for $\mathbb{P}^n$ is the set $S = \{1, t, t^2, \ldots, t^n\}$.

Fact: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a spanning set for a subspace H of a vector space V. Then (if $H \neq \{\mathbf{0}\}$) some subset of S is a basis for H.